*The Elasticity of Rubber Balloons and Hollow Viscera.**

By Prof. W. A. OSBORNE, with a Note by W. SUTHERLAND.

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Introductory Theory.

In an elastic balloon the relation between the internal excess pressure and the tension of the wall can be readily calculated if we assume that the balloon is spherical and that the material is homogeneous and of negligible weight. If we suppose the balloon divided into two hemispheres by a plane horizontal partition, the area of this partition will be πr^2 and the downward force on the upper surface due to the excess pressure p will be $\pi r^2 p$. The balloon wall meets the partition at right angles along a length $2\pi r$. Hence if T is the tension in the wall, the upward force exerted by this tension on the partition is $2\pi r\mathrm{T}$. But as these two forces must be equal we have

$$\pi r^2 p = 2\,\pi \mathrm{T} r, \quad \text{so that} \quad p = 2\,\mathrm{T}/r. \tag{1}$$

When such a balloon is filled without stretching the wall the pressure inside is equal to the prevailing atmospheric, and the radius r_0 may be termed the initial radius. If we assume that the balloon is perfectly obedient to Hooke's law, then

$$\mathrm{T}_1 = \mathrm{K}\,(r_1 - r_0)/r_0\,;$$

but from (1) we learn that

$$p_1 = 2\,\mathrm{T}_1/r_1\,;$$

hence, by substitution, $\quad p_1 = 2\,\mathrm{K}/r_0 - 2\,\mathrm{K}/r_1,$

or

$$r_1\left(\frac{2\mathrm{K}}{r_0} - p_1\right) = 2\,\mathrm{K}. \tag{2}$$

That is to say, the pressure will increase with radius asymptotically to $2\,\mathrm{K}/r_0$, and if we plot radius against pressure we shall obtain a rectangular hyperbola.

The original object of the following research was to investigate the elastic behaviour of various hollow viscera. Before doing so I decided, however, to carry out a number of experiments with rubber balloons, using the pressures found with varying radii as fundamental data.

* This research was completed before I became aware, from a reference in Boruttau's 'Lehrbuch der Medizinischen Physik,' that a similar investigation had been carried out by R. du Bois-Reymond, and the results published in the 'Festschrift für Rosenthal.' On obtaining the latter, I found sufficient difference in the treatment of the subject to warrant publication of this. [See also A. Mallock, 'Roy. Soc. Proc.,' vol. 49, p. 458.]

Methods.

The balloons were of the common variety sold as children's toys, and were of varying sizes. A balloon was firmly tied to a glass capillary cannula and held vertically in a glass flask which was immersed up to the neck in an Ostwald thermostat (fig. 1). The temperature of the latter was kept (with a maximum variation of 0°·1 C.) at 35°·5 C. The glass cannula was connected by means of fine-bore pressure-tubing to one limb of a capillary **T**-piece, a second limb of which led to a water manometer, whilst the third limb was connected with a burette for admitting measured volumes of air. The connecting tubes were made as short and of as narrow bore as possible so that the contained volume of air could be neglected in calculation. The burette at its lower end was connected with a levelling tube containing mercury, and at its upper end had a three-way tap. In one position of the tap a sample of air of definite volume and at atmospheric pressure could be taken from the outside air; on the tap being turned this air could be driven through the connecting tubes into the balloon, the mercury being accurately brought to the beginning of the bore of the tap. Conversely, the balloon could be deflated in measured decrements by the same burette. The water manometer consisted of a straight glass tube of 3 mm. bore firmly tied to a vertical scale, and connected at its base with a shorter vertical tube on which was a mark. By means of a three-way tap, capable of connecting the manometer either with the outside air or with an elevated reservoir of water, the level of the water in the shorter limb could be brought to its mark, allowing direct readings to be made from the scale as well as preventing change in the volume of the tube system.

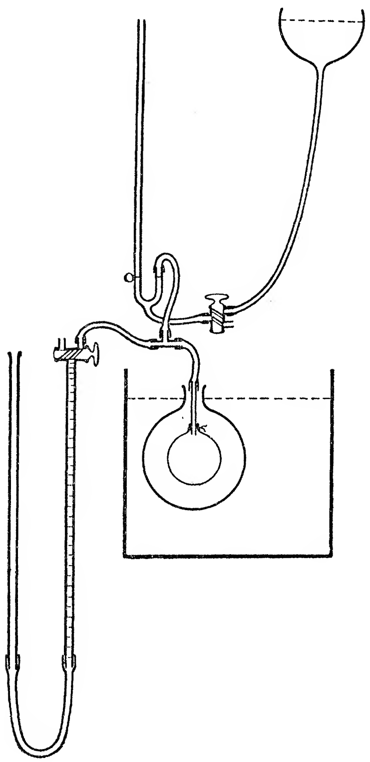

Fig. 1.

The radius of the balloon was calculated from the volume of the air admitted by the usual formula. This involved two assumptions, first that the balloon was spherical, and secondly, that the volume of the enclosed air was the same as that of the air admitted at atmospheric pressure. With the exception of the early stages of inflation and last stages of deflation, when the radius approached the initial value, the balloon could be regarded as a true sphere. As, further, the greatest pressure within the balloon was always a negligible fraction of the prevailing atmospheric, I have not thought it necessary to make any calculated correction as to the volume of the contained air.

When experiments were performed on a hollow viscus, some water was placed in the partially immersed flask, and a few drops placed in the interior of the viscus itself so that the air within and without should be saturated with water vapour.

Experiments on Balloons.

When air was admitted in measured increments to a fresh balloon, and the reading taken a definite time (three minutes) after entrance of each increment, it was found that the pressure rose quickly to a maximum and then on continued inflation fell slowly. This is typically exemplified in the experiment illustrated graphically in fig. 2.

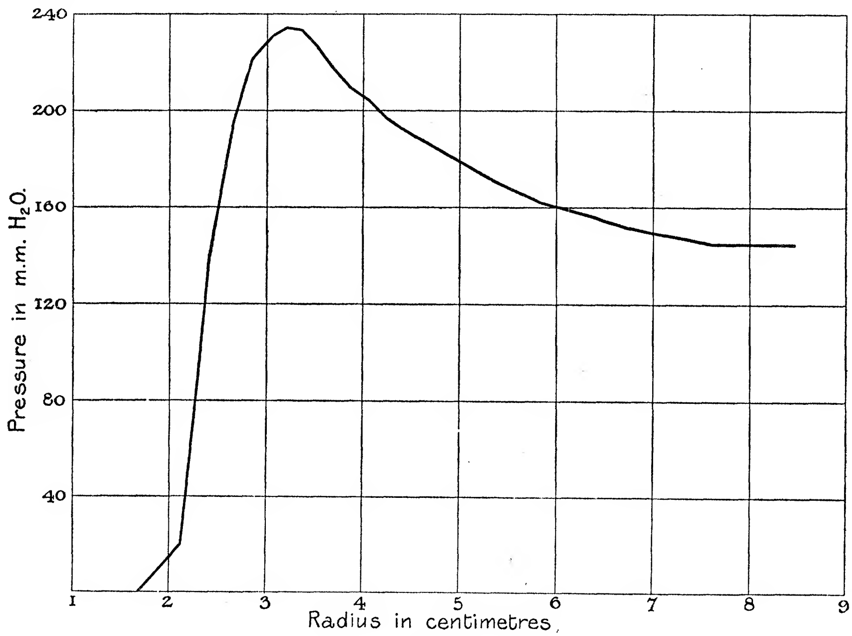

FIG. 2.

It will be seen from this experiment that, over a considerable range, two values of radius can be given for each value of pressure. This can be demonstrated as a class experiment in the following way: Two balloons of equal dimensions are tied to two limbs of a T-tube and inflated by the third limb, which can be closed by a tap. As a rule one of the balloons inflates well, the other remaining small. On closing the tap in the inflating tube the contents of one balloon can be discharged into the other by squeezing with the hand. If the air be worked backwards and forwards a few times to equalise the "history" of each, it will be found that if the balloons are approximately equal in volume they will remain so for a few seconds, in a state of unstable equilibrium, and then one of the balloons will partially deflate itself into the other. The balloon which is now the larger, if squeezed until its volume is slightly less than that of the other and then let go, will continue to deflate until equilibrium is reached.

These experimental results appeared to be utterly at variance with what was deducible from the theory of a perfectly elastic balloon. Amongst the many articles dealing with the elasticity of rubber to which I had access, I found one which promised to throw some light on my results. O. Frank* assumes a somewhat modified Hooke's law. According to him the pressure dP in a sample of section q and length x associated with a shortening dx is given by the formula

$$dP/q = E\,dx/x,$$

in which unit initial length and unit initial cross sectional area are not considered, but length and area such as they are when the change dx is produced. If x_0 is the original length and x_1 the final, he calls $\Lambda = (x_1 - x_0)/x_0$ the specific extension. For the total tension in a strip of unit width and of initial thickness z_0 his final result on p. 608 can be written

$$T = 2E\Lambda\frac{z_0}{(1+\Lambda)^2}.$$

Substituting this value of T in equation (1), we get

$$p_1 = \frac{4E\Lambda z_0}{r_1(1+\Lambda)^2}.$$

But in the case of the balloon the specific extension $\Lambda = (r_1 - r_0)/r_0$, therefore

$$p_1 = 4Ez_0r_0\frac{r_1 - r_0}{r_1^3}.$$

According to this equation the pressure in an inflating balloon will rise to a maximum when $r_1 = \frac{3}{2}r_0$, and will approach asymptotically to zero when r_1 increases indefinitely. But I may say at once that this approach to zero pressure is never given in balloon experiments, so that Frank's analysis fails to explain the results obtained. One may indeed state *a priori* that as

* 'Annalen der Physik,' vol. 21, p. 602, 1906.

investigations on elasticity are generally confined to substances where the maximum extension is always a small fraction of the initial length, and as Frank's experiments did not follow rubber further than linear extensions to double the initial, it would be almost idle to expect that laws deduced from these experiments could be applicable to the large and two dimensional stretchings of an inflated balloon.

The difficulty in explaining the rise of pressure and the subsequent partial fall on inflation is, I believe, more apparent than real. This crest is due, I take it, to a disturbing factor which, for lack of a better name, may be called initial rigidity. This view is supported by the following facts:—

1. If s fresh balloon is inflated, so that the pressure is anywhere on the rise or fall of the crest, it will be found that the pressure does not remain at a constant value, but tends to fall. In fact, to obtain a graph such as fig. 2, the convention had to be adopted of reading the pressure after a given interval of time—3 minutes. But the fall had by no means stopped when the reading was taken, and could be detected even some hours after inflation. An attempt to register the pressure after a long interval of time when no further fall might be expected, failed owing to the fact that some of the air diffused out, as was proved by deflating the balloon in measured decrements.

2. If a balloon is inflated a second time (care being taken that the elastic limit has not been reached in the first inflation) the crest is always less pointed than in the first inflation. A third inflation gives a more obtuse convexity than the second, and so on. The longer a balloon remains collapsed the steeper is the rise and fall of pressure on inflation. This is particularly marked if the collapsed balloon is exposed to light.

3. When an inflated balloon is deflated in measured decrements and the corresponding pressures recorded, in the vast majority of cases the pressure falls to zero without any rise being manifested. I obtained this pronounced hysteresis constantly in my earlier experiments, and was inclined to look upon it as the invariable behaviour of a balloon during deflation. Fig. 3 gives graphs for two typical instances.

But a rise of pressure may be obtained on deflation if certain conditions are fulfilled. The rubber must be in good condition, the inflation should not be taken far past the maximum pressure, and the return by deflation should be carried out at once. The rise of pressure, however, is never more than a few millimetres of water. The better condition the rubber is in the blunter is the inflation crest and the less abrupt is the deflation fall of pressure. Conversely, the more the rubber has been exposed in a deflated state to light the sharper is the crest and the more abrupt is the deflation fall.

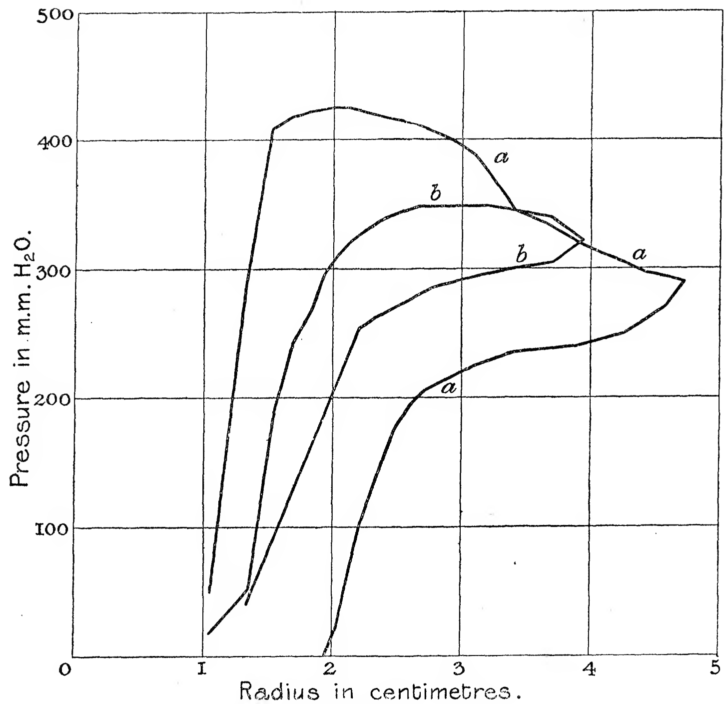

Fig. 3.

I may mention in this connection that if inflation be carried out immediately after deflation the rise of pressure does not follow the same gradient as the deflation fall. It is much steeper, and a crest may be obtained. An illustrative specimen is the following (fig. 4):—

It is easy to demonstrate, however, that the more a balloon is inflated and

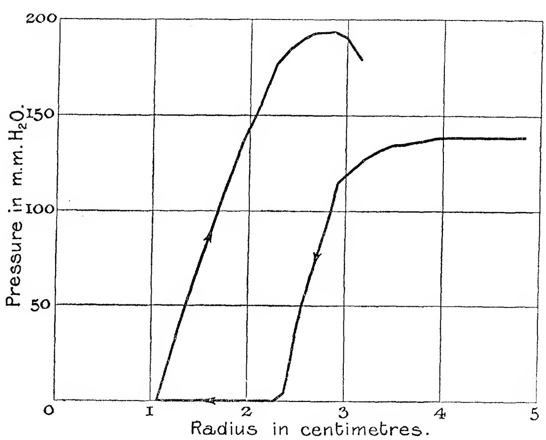

Fig. 4.

deflated, provided that the elastic limit is not approached too closely, the nearer does the inflation pressure gradient approach the deflation. We may regard this as due to the partial removal of the disturbing initial rigidity.

4. If a balloon be inflated until the pressure, after the usual crest, falls and tends to remain constant, and be kept inflated for some time, say 24 hours, and then be rapidly deflated and once more inflated in measured increments, the graph displays no crest and may be a true hyperbolic curve. The following experiment illustrates this important fact:—

A balloon was inflated until the pressure ceased falling, and was kept inflated in the thermostat for 24 hours. It was then rapidly deflated and the usual inflation by the burette commenced. On plotting pressure against radius (fig. 5), I was struck by the regularity of the graph, and recollecting that a balloon of perfect elasticity would give a rectangular hyperbola,

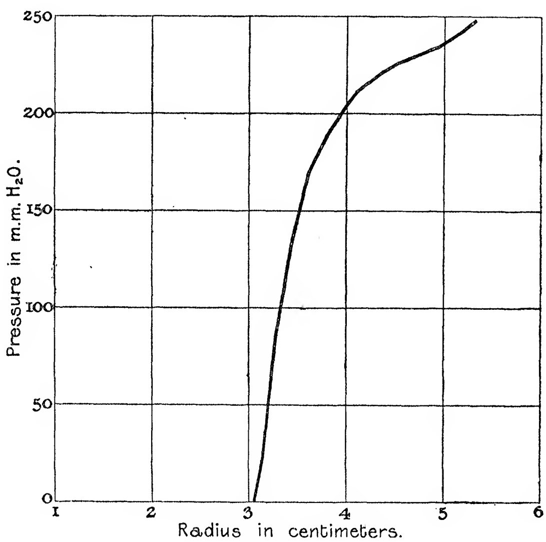

FIG. 5.

proceeded to ascertain if such were the case here. If this were a rectangula hyperbola, the asymptotes being parallel to the co-ordinate axes, it ought to satisfy the equation $(r-a)(p-b) = c$.

To calculate a and b I used the ordinary three-point method. The value for a was found to be 2·8, that of b 287·8.

From radius $= 3{\cdot}29$ to radius $= 4{\cdot}37$ the product $(r-a)(b-y)$ is a constant. To illustrate this graphically we can plot $b-y$ against the reciprocal of $r-a$, and should obtain a straight line passing through the origin. This is shown in fig. 6.

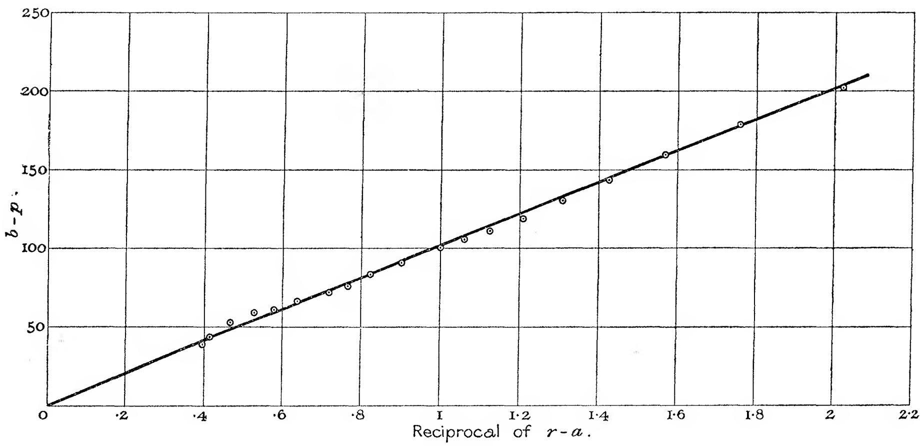

FIG. 6.

As another instance of the applicability of this equation to a balloon, the deflation values already given in fig. 3 may be cited. Calculation by the three-point method gives here $a = 2·03$, $b = 263$.

Conclusive as these values are that the rubber balloon, when initial rigidity is removed, follows the equation $(r-a)(p-b) = c$, it will be at once obvious, from the values of a and b found here, that this is certainly not the behaviour of a perfectly elastic substance giving equation (2). For one thing, the value for a is far removed from zero and is suggestively close to that of the initial radius in the two cases investigated. I abandoned the theoretical analysis of my results at this stage, and handed over my data on balloons and on bladders to Mr. William Sutherland, who has kindly complied with my request to comment upon them (see p. 497 below).

Rubber Balloons at the Elastic Limit.

In the course of this research a curious result was obtained with every balloon which I inflated beyond the elastic limit. I invariably found that, before the balloon burst, the pressure, over a considerable range, was a linear function of the volume. Of the many instances obtained I will pick out two, one giving a close approximation to a straight line on plotting volume against pressure (fig. 7).

One of the more divergent types is that given in fig. 8, which is a continuation of the same experiment as fig. 2.

As a rule, the straight line rises abruptly, producing discontinuity in the graph.

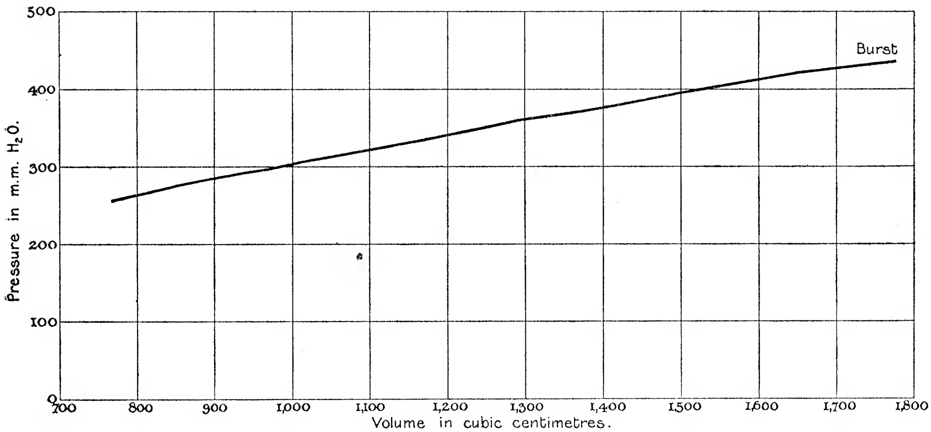

FIG. 7.

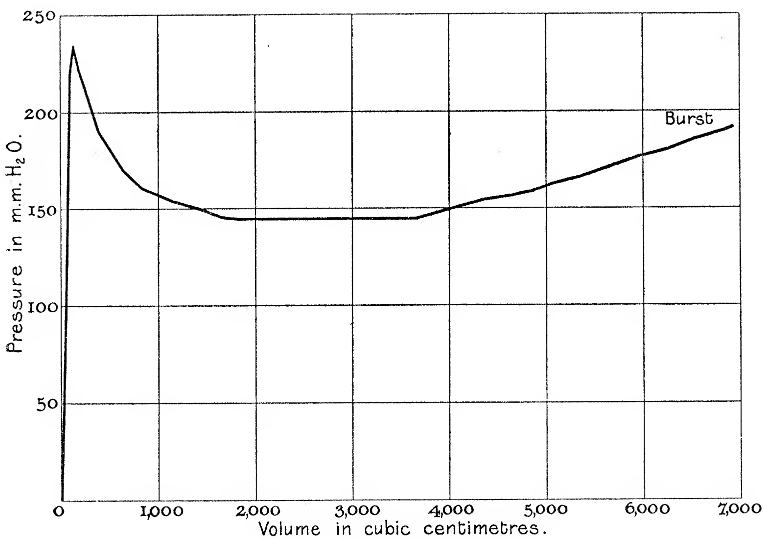

FIG. 8.

Experiments with Hollow Viscera

In these experiments attention was confined chiefly to the bladder, as its shape approaches more closely to the spherical than other viscera. Experiments on lungs proved impossible, owing to the remarkably low bursting pressure of the superficial air cells. A number of observations were made with bladders taken from the recently killed animal, but the erratic behaviour of the living muscular tissue did not allow of a definite pressure being assigned to any stage of inflation. Consistent results could only be obtained by experimenting with bladders some time (24 hours) after the death of the animal.

Fig. 9 shows the results of an experiment with the bladder of a large Newfoundland dog 24 hours after death. As with the balloon, I anticipated that here a hyperbolic curve was present, and calculated by the three-point method the value for a to be 0·071, b to be 179. Here it will be seen that from radius 1·93 to radius 2·88 a distinct approximation to a rectangular

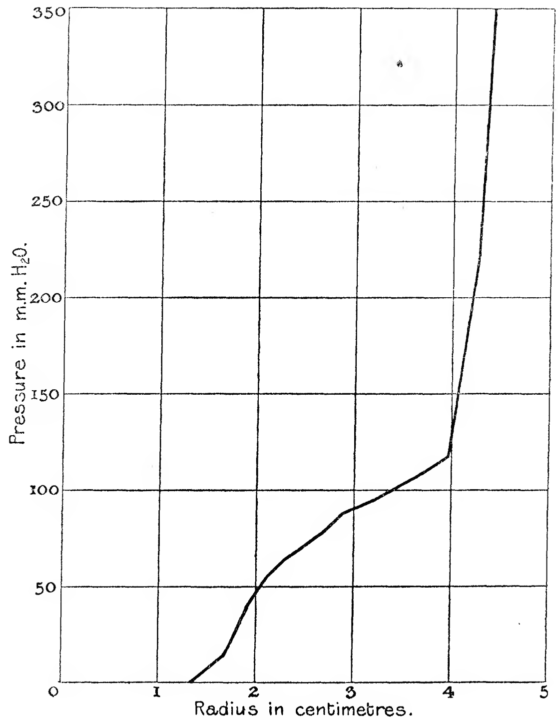

Fig. 9.

hyperbola is manifest. But even here, though a can be made zero without appreciably altering the constancy of c, the value for b likewise does not allow us to apply to this bladder the formula for a perfectly elastic substance.

A number of bladders of various animals were investigated. I give here the results obtained with the bladders of two monkeys and a cat (fig. 10).

It must be remembered that the elastic tissue of a viscus is not a homogeneous membrane, but a web of elastic fibres with a variable amount of inextensible white fibres intermixed. This fact must always complicate physical investigations on the elasticity of animal membranes, even if the isolated elastic fibres present obeyed some simple physical law.* When we

* A research on the elastic constants of the *ligamentum nuchæ* is at present being conducted in my laboratory.

consider the complex æolotropism of a visceral wall, it is indeed surprising that approximations to uniform behaviour, such as are illustrated in fig. 10, should be shown at all.

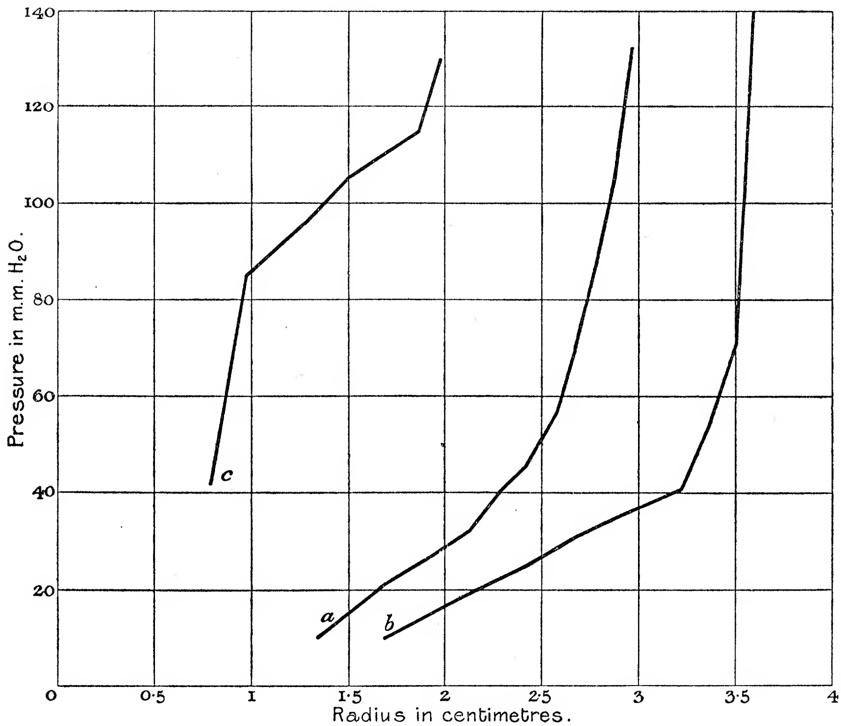

FIG. 10.

R. du Bois-Reymond has conjectured that in hollow viscera the pressure may fall with increasing volume. I may state at once that I have never found this. What sometimes does happen (and to this Du Bois-Reymond's statement is possibly due) is that, on extensive inflation, one of the coats of the organ may give way and lead to a marked drop in pressure. The suddenness of the drop will always indicate the true nature of the fall, and if the organ be now deflated and then inflated again, a consistent rise of pressure will be obtained. Moreover, as I have endeavoured to show, a fall of pressure on continued inflation is only found in balloons manifesting initial rigidity, and such initial rigidity is altogether absent from animal membranes kept moist.

A bladder always displays some hysteresis on deflation, but I have found that this hysteresis can be made negligibly small—(1) if the elastic limit is not approached too closely; (2) if the inflation and deflation are carried out by very small increments and decrements respectively; and (3) if on deflation some time be allowed to elapse at each stage before reading the pressure, as this always tends to rise somewhat. When the elastic limit of a bladder is reached, the gradient of the pressure rise is very steep and the rise is not a linear function of the volume.

There is always a danger that in investigations on elasticity one may forget that the viscus in question in the living animal is supplied with reactive muscle, and that only when this muscle is fully inhibited can the pure physical elasticity of the walls play a predominating part. It is a mistake to describe the flow of blood in the systemic arteries as a flow of liquid in elastic tubes. Such is certainly the case in the aorta, and possibly in the larger arteries, but in the arterioles and smaller arteries only when the muscle is fully inhibited or killed. To describe the circulation as occurring through a system of muscular tubes, with some elastic tissue aiding the muscles, would be more accurate. Similarly with the bladder and other hollow viscera (except the lung), the elastic tissue acts merely as an adjuvant to the muscle, economising the work of the latter; but it is the muscle which plays the preponderating part in determining the tension of the visceral wall.

Conclusions.

1. When initial rigidity is present in a rubber balloon, the pressure on inflation rises rapidly at first, then falls, and tends to remain at a constant value until the elastic limit is reached.

2. Such a balloon on deflation displays a marked hysteresis. Only rarely will the pressure rise on deflation.

3. If initial rigidity be abolished by keeping a balloon inflated some time and then rapidly deflating, the pressure on a new inflation rises consistently. On plotting pressure against radius in such cases a rectangular hyperbola may be obtained, satisfying the equation

$$(r-a)(p-b) = c,$$

where a is close in order of magnitude to the initial radius, and b is a constant greater than p. The behaviour of such a balloon is, however, far removed from that of a sphere of perfectly elastic and isotropic material.

4. When the elastic limit is reached in a rubber balloon the pressure is a linear function of the volume.

5. Hollow viscera approximately spherical, such as the bladder, do not display initial rigidity, and never give a fall of pressure with increasing volume. When the elastic limit is reached, the pressure is not a linear function of the volume.

6. In the bladder of a large dog, giving sufficient range between the assumption of globular form and the elastic limit to allow analysis of the graph of pressure against radius, it was found that the equation

$$(r-a)(p-b) = c$$

was followed. In this case a was practically zero; but like the rubber balloon the behaviour was not that of a perfectly elastic and isotropic substance.

Note on the foregoing Paper by W. SUTHERLAND.

From the purely physical point of view the simplest way to prepare for a theoretical interpretation of experiments such as these is to fix attention in the first instance on tension per unit area.

Let the tension per cm.² be t in the balloon or bladder which has radius r and thickness z. Let initial values of these, when $t = 0$, be r_0 and z_0. Consider the equilibrium of a hemisphere. It experiences a pull $2\pi rzt$ from the other hemisphere. But on account of the excess p of the pressure inside the sphere over that outside the hemisphere is subject to a thrust $\pi r^2 p$; thus

$$\pi r^2 p = 2\,\pi rzt \quad \text{or} \quad pr^2 = 2\,rzt. \tag{1}$$

If, as in studying the surface tension of bubbles, we fix attention on zt, the total tension across unit width of cross-section of the bounding wall, and call it T, we have

$$p = 2\,\mathrm{T}/r. \tag{2}$$

According to Hooke's law, we write

$$t = \mathrm{E}\,(r-r_0)/r_0, \tag{3}$$

where E is a modulus of elasticity appropriate to the conditions of the experiment, which in the present case are equal tensions in two dimensions and no external stress in the third dimension. For substances such as rubber and most organic tissues which have a compressibility, small in comparison with their deformability, E for small strains is twice the ordinary Young's modulus for small strains. But when large strains are used, as in these experiments, E can no longer be treated as a constant. It is a function of the strain. This appears when we compare Prof. Osborne's formula $(p-b)(r-a) = c$ with (1) and (3), after elimination of z by the relation $r^2 z = r_0^2 z_0$ expressing incompressibility, for we get

$$p = \frac{b\,(r-a)+c}{r-a} = \frac{2\,\mathrm{E}z_0\,(r-r_0)\,r_0}{r^3}. \tag{4}$$

This makes E a complicated function of r.

In the experiments with the dog's bladder, a is nearly 0, so that this takes the simpler form $E = \frac{r^2(br+c)}{2z_0(r-r_0)r_0}$, which is still too awkward for interpretation. But to connect the results for the tissue of dog's bladder with those for other tissues the modulus of elasticity E can be regarded from a different point of view. In experiments on dead muscle, for instance, the muscle is stretched by different weights, the amount of stretching produced by each being recorded. As the muscle is lengthened its cross-section is diminished, but, as a rule, no account is taken of this fact. This is because more interest is taken in the behaviour of the muscle as a whole, or of a single representative muscle fibre, than in the intensity of the tension or the tension per cm.^{2} of cross-section of the muscle. For the gastrocnemius of the frog stretched by amounts $l-l_0$, by weights w up to 95 grammes, C. Henry has shown* that the following formula holds:

$$l-l_0 = 6{\cdot}55 \log(1+w/6{\cdot}10), \tag{5}$$

$l-l_0$ being expressed in mm. and w in grammes weight. For other tissues with a wide range of elastic properties, A. Goy† finds the same formula to apply with appropriate values in place of 6·55 and 6·10. But the physical explanation given for (5) by Henry is not sound, as he interprets $1+w/6{\cdot}10$ in the form $(6{\cdot}10+w)/6{\cdot}10$ to mean that there is at the beginning a tonus of the muscle equivalent to a weight 6·1 grammes. If there is stress in the muscle at the beginning it must be self-equilibrating, and it is not correct mechanics to fix upon one part of this internal stress, called the tonus, and treat it as a sign of a not otherwise demonstrable external force denoted above by 6·1. But, guided by the success of (5), we can arrive at a simpler formula which is capable of legitimate and easy physical explanation. Let us suppose that the elongation $l-l_0$ caused by w is related to w by the following equation:

$$(l-l_0)/w = a-b(l-l_0), \tag{6}$$

where a and b are constants for a given tissue.

This means that the average elongation caused by unit weight, that is to say $(l-l_0)/w$, diminishes with increasing w in such a way that the diminution is linear in the total elongation produced by w. When $b = 0$, we have the usual Hooke's law for small strains. It is possible to give a theoretical molecular explanation of (6), though it would not be appropriate here. In the case of the frog's gastrocnemius, for values of w from 30 to 95 grammes, it gives the elongation $l-l_0$, with a maximum error of 1·6 per cent., and from 0 to 30 grammes with a maximum error of 16 per cent., the

* 'Compt. Rend.,' vol. 162, 1906, p. 729.

† *Ibid.*, p. 1158.

corresponding error for (5) being 17 per cent. But it is probable that in neither case are these really errors of formula, because with the smaller weights there is liability to considerable experimental uncertainty while "taking up the slack" of the specimen.

It is interesting to see how the type of formula (6) applies to Prof. Osborne's experiments on the bladder of a dog. We must treat the experimental facts so that they are as similar as possible to those of muscle. If we return to (1) we see that πpr^2 corresponds with w the weight used to stretch muscle, although it stretches the bladder wall in two directions at right angles to one another. The chief effect of this stretching in two directions is to replace E as measured on a strip cut from the bladder wall and stretched only in one direction by 2 E. From the experiments we get

$$\frac{r-r_0}{\pi pr^2} = 0{\cdot}00121 - 0{\cdot}000318\,(r-r_0), \tag{7}$$

with the following comparison:—

r	1·336	1·680	1·927	2·121	2·285	2·429	2·672	2·878
p exper. ...	0	14	40	55	63	68	77	87
p calc.	0	35	50	58	64	68	76	82

r	3·220	3·637	3·959	4·252	4·493	4·655	4·856
p exper. ...	95	107	117	217	410	560	620
p calc.	95	116	142	182	243	318	533

At the two lowest pressures after 0 the discrepancy between calculation and experiment is large, but can plainly be ascribed to the taking up of slack in the experiments. The formula fits the facts satisfactorily over the very great elongation from $r = 2{\cdot}121$ to $r = 3{\cdot}220$. Beyond that the formula ceases to give the connection between p and r in a useful manner, but on that account it by no means loses its physical significance. If we write (7) in the form

$$p = (r-r_0)/\pi r^2\,\{0{\cdot}00121 - 0{\cdot}000318\,(r-r_0)\}, \tag{8}$$

we see that for values of $r-r_0$ greater than 3 the difference $0{\cdot}00121 - 0{\cdot}000318\,(r-r_0)$ becomes small compared with either $0{\cdot}00121$ or $0{\cdot}000318\,(r-r_0)$. Hence a small error in r produces a much larger relative error in p. With this fact in view it appears that (7) gives a good account of the physical happenings in the wall of the bladder during the large elongations up to $r = 4{\cdot}856$.

The form (7) can be applied to the experiments on a deflated rubber balloon, but not to those on an inflated.